CONNECTIVITY AND NETWORKS

We begin with the definition of a few symbols, two of which can cause great confusion, especially when hand-written. Consider a graph G.

$\delta(G)$ the degree of the vertex with smallest vertex degree; the **minimum degree** of G

$\Delta(G)$ the degree of the vertex with the largest vertex degree; the **maximum degree** of G

k(G) the **number of components** in G. G is connected if and only if k(G) = 1.

$\kappa(G)$ the **vertex connectivit**y of G, commonly called the **connectivity** of G. This is the minimum number of vertices whose removal from G results in either a disconnected graph or trivial graph.

$\kappa_1(G)$ the **edge connectivity** of G. This is the minimum number of edges the removal of which from G will result in either a disconnected graph or trivial graph.

$G - v$ for a given vertex $v \in V(G)$, this is the graph obtained from G by removing the vertex v and all edges incident on that vertex.

$G - e$ for a given edge $e \in E(G)$, this is the graph obtained from G by removing the edge e. This does not remove the vertices upon which the edge is incident.

The student will note that the two symbols k(G) and $\kappa(G)$ have the possibility for causing a lot of confusion, especially when this instructor normally writes the term k(G) when he intends to write $\kappa(G)$. This will be seen below in the definition of a cut vertex.

Definition: A vertex is a **cut-vertex** of a graph G if its removal from G generates a new disconnected component. Put another way, a vertex $v \in V(G)$ is a cut vertex if $k(G - v) > k(G)$, that is the number of components in G with v removed is greater than the number in G.

The source of confusion in the terminology can be seen by noting that if v is a cut vertex, then it is most likely (though not required) that $\kappa(G - v) \leq \kappa(G)$. We shall show this later.

Definition: A **bridge** of a graph G is an edge e such that $k(G - e) > k(G)$; that is to say that the removal of the edge creates a new disconnected component of G.

Definition: A **block** is a nontrivial connected graph with no cut vertices.
Definition: A **block of a graph G** is a subgraph of G that is itself a block and which is maximal with respect to that property.

As an example, let's look at the following figure and find the cut vertices and bridges, etc.

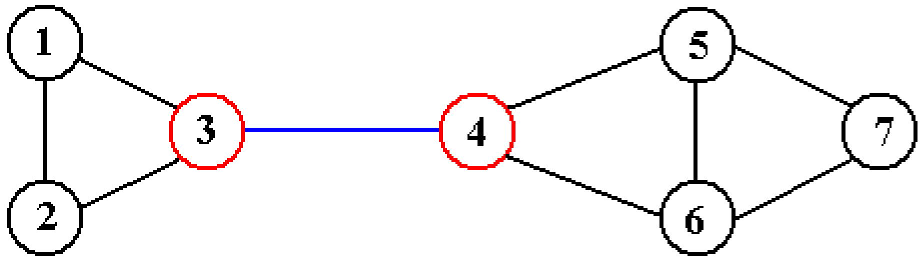

Figure 26: A Graph G with Two Blocks

Formally V(G) = {1, 2, 3, 4, 5, 6, 7}
E(G) = { (1, 2), (1, 3), (2, 3), (3, 4), (4, 5), (4, 6), (5, 6), (5, 7), (6, 7) }

We begin by noting the degree of each vertex in the above graph:
$d_1 = 2$, $d_2 = 2$, $d_3 = 3$, $d_4 = 3$, $d_5 = 3$, $d_6 = 3$, and $d_7 = 2$. Note that the degrees of these vertices sum to 18, twice the number of edges (as expected).

The degree sequence of G is (3, 3, 3, 3, 2, 2, 2) as the degree sequence of a graph presents the degrees of the vertices in non-increasing order.. We see that $\Delta(G) = 3$ and $\delta(G) = 2$. This is obvious from reading the degree sequence, but can be seen also from examining the graph.

Note that k(G) = 1 as the graph is connected. Note that, by coincidence, $\kappa(G) = 1$ also as the removal of either vertex 3 or vertex 4 will cause the graph to break into two disconnected components. Thus each of vertex 3 and vertex 4 is a **cut-vertex**. The next figure shows the graph $G - v_3$.

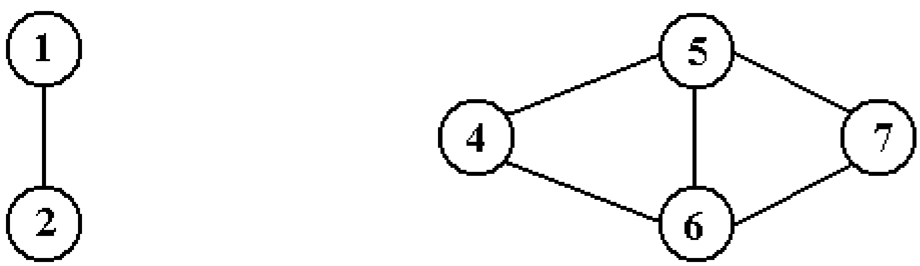

Figure 27: The Graph G – v_3

In this case, the edge connectivity of the graph is also one; $\kappa_1(G) = 1$. This can be seen by noting that the removal of edge (3, 4) from the original graph G will disconnect the graph.

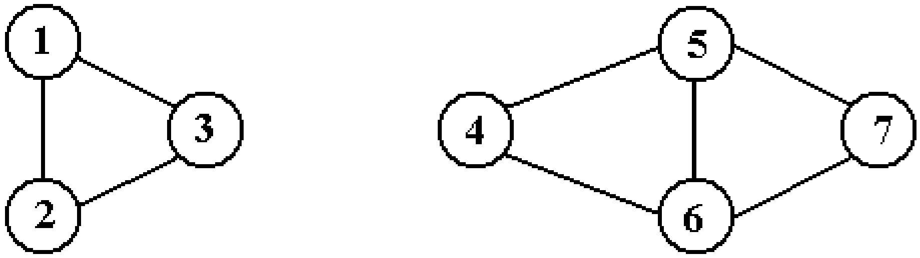

Figure 28: The Graph G – (3, 4)

We now go searching for the blocks of G. To do this precisely, we should examine figure 26, but we refer to figure 28 which actually shows the two blocks: {1, 2, 3} and {4, 5, 6, 7}. Note that the subgraph induced by vertices {4, 5, 6} is not a block. True, the graph induced by {4, 5, 6} is a K_3 (triangle) and is a nontrivial connected graph with no cut vertices, but it is not maximal with that property as one can add vertex 7 to that set and still have a block. So, we now have the block as {4, 5, 6, 7}. Note that vertex 4 becomes a cut vertex only when we consider vertex 3, so the set {3, 4, 5, 6, 7} forms a subgraph that does contain a cut vertex.

Let's make this important point another way. Each of vertices 3 and 4 is a cut vertex in the graph G. Within the subgraph induced by the vertex set {4, 5, 6, 7}, vertex 4 is not a cut vertex as its removal will not cause that subgraph to become disconnected.

We now state and partially prove one of the basic theorems on connectivity.
Theorem 21: For any graph G, $\kappa(G) \leq \kappa_1(G) \leq \delta(G)$.
Note: To show that $\kappa_1(G) \leq \delta(G)$, take the vertex of minimum degree in G and remove all its edges. This isolates the vertex and causes the graph to be disconnected.

In our example, $\kappa_1(G) = 1$ and $\delta(G) = 2$.

We now state some theorems related to connectivity and actually prove a few.
Theorem 22: A vertex *v* of a connected graph G is a cut-vertex of G if and only if there exist vertices distinct vertices *x* and *y* ($x \neq v$, $y \neq v$) such that *v* is on every *x*-*y* path of G.
Proof: Let *v* be a cut-vertex of G so that the graph G – *v* (G with the vertex *v* removed) is disconnected. Let *x* and *y* be vertices in different components of G – *v*, then there are no *x*-*y* paths in G – *v*. However, G is connected, so there is at least one *x*-*y* path in G. Therefore, every *x*-*y* path in G contains the vertex *v*.

Conversely, assume that there exist vertices *x* and *y* in G such that a vertex *v* lies on every path between them. Then, there are no *x*-*y* paths in G – *v*, implying that G – *v* is disconnected, and that *v* is a cut-vertex of G.

Theorem 23: An edge *e* of a connected graph G is a bridge of G if and only if there exist vertices *x* and *y* such that the edge *e* is on every *x*-*y* path of G.

Theorem 24: An edge *e* of a graph G is a bridge of G if and only if *e* is on no cycle of G.

Recalling that a block is a non-trivial graph without cut vertices, we have the following.
Theorem 25: A graph G with $n \geq 3$ vertices is a block if and only if every two vertices of G lie on a common cycle of G.

Recalling that a graph is called ***r*-connected** if $\kappa(G) \geq r$, that is, it requires the removal of at least *r* vertices to cause the graph to become either disconnected or trivial, we have this.
Theorem 26: Let G be a graph with $n \geq 2$ vertices, and let *r* be an integer such that $0 < r < n$. If $d(v) \geq \lceil (n + r - 2) / 2 \rceil$ for every vertex $v \in V(G)$, then G is *r*-connected.

Corollary 27: Let G be a graph with $n \geq 3$ vertices. If $d(v) \geq \lceil n / 2 \rceil$ for every vertex $v \in V(G)$, then G has no cut-vertices.
Proof: This is the above theorem with $r = 2$ and $n \geq 3$ to avoid trivial cases. The graph is 2-connected, implying that there is no single vertex the removal of which will disconnect the graph. Hence the graph has no cut-vertices.

One should note that the implication of the above statement is that G is a dense graph. The reasoning is quite simple. If $d(v) \geq \lceil n / 2 \rceil$ for every vertex $v \in V(G)$, then the sum of the vertex degrees is greater than $n \bullet \lceil n / 2 \rceil \geq n^2 / 2$, so (by Theorem 3), $2 \bullet m \geq n^2 / 2$, and G is dense.

One should be cautious not to infer that dense graphs lack cut-vertices; the only valid conclusion is that they are less likely to contain cut-vertices. The next figure shows that dense graphs can contain cut-vertices. It is a (6, 11)-graph that is a dense graph (as $6^2/4 = 36/4 = 9$). The vertex labeled with the asterisk is a cut-vertex.

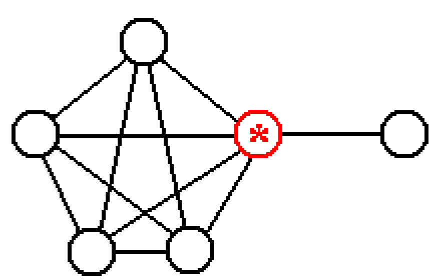

Figure 29: Dense Graph with a Cut-Vertex

We now generalize the idea of connectivity (also called 1-connectivity) with the following two theorems that can prove quite useful for the study of networks. We begin with a few definitions, them move on to the two theorems.

Definition: A set S of vertices (or edges) of a graph G is said to **separate** two vertices *u* and *v* of G if the removal of the elements of S from G produces a disconnected graph in which the vertices *u* and *v* lie in different components.

Recall that a path between two vertices *u* and *v* can be described as a sequence of distinct vertices $x_0, x_1, x_2, \ldots, x_p$, such that $u = x_0$, $v = x_p$, and that every vertex is adjacent to the one following it in the sequence. The path is created by following the edges, so one starts with (x_0, x_1) and continues through the sequence until finishing with (x_{p-1}, x_p). With this in mind, we state the next definitions.

Definition: An **internal vertex** of a *u*-*v* path P is any vertex of P that is not either *u* or *v*.

Definition: Two *u*-*v* paths are **internally disjoint** if they have no internal vertices in common.
For example, let P_1 and P_2 be two paths between vertices *u* and *v*. It should be obvious that each of the paths P_1 and P_2 contain both the vertices *u* and *v*. If these two vertices are the only vertices in common to the two paths, then specifically they share no internal vertices and the two paths would be called internally disjoint.

Definition: A collection $\{P_1, P_2, \ldots, P_k\}$ of paths is called **internally disjoint** if each pair of paths is internally disjoint.

Theorem 28 (Menger): Let *u* and *v* be nonadjacent vertices in a graph G. Then the minimum number of vertices that separate *u* and *v* is equal to the maximum number of internally disjoint *u*-*v* paths in G.

Theorem 29 (Whitney): A nontrivial graph G is *r*-connected if and only if for each pair *u*, *v* of distinct vertices there are at least *r* internally disjoint *u*-*v* paths in G.